Computational Logic Lab VI

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Craig Interpolation's Lemma. The property of interpolation is an important property of first order and propositional logic. It basically says that if we have an implicational tautology[1], then there exists a formula (an interpolant) that is logically implied by that tautology's antecedent and that logically implies the tautology's consequent. In this session we will introduce its propositional version (whose proof is much simpler):

Theorem 1 (Propositional interpolation)**.** *If* $\models \varphi \Rightarrow \psi$*, then there exists a formula* χ*, called* interpolant*, with* $Atom(\chi) \subseteq Atom(\varphi) \cap Atom(\psi)$*, s.t.* $\models \varphi \Rightarrow \chi$ *and* $\models \chi \Rightarrow \psi$*.*

To prove Theorem 1, we will prove an equivalent statement, by induction on $\#(Atom(\varphi) \setminus Atom(\psi))$, namely:

Lemma 1. *If* $\models \varphi \wedge \psi \Rightarrow \bot$*, then there exists an interpolant* χ *with* $Atom(\chi) \subseteq Atom(\varphi) \cap Atom(\psi)$*, s.t.* $\models \varphi \Rightarrow \chi$ *and* $\models \psi \Rightarrow \neg\chi$*.*

The proof of Proposition 1 is *constructive*, and sketches a procedure for computing interpolants.

Exercises. Consider now the following exercises:

1. Show that Theorem 1 is a corollary of Lemma 1.
2. Provide a pseudo-code specification of $\textsc{Craig}(\cdot, \cdot)$, the algorithm that computes the interpolants of formulas φ and ψ.
3. Compute interpolants for the tautology

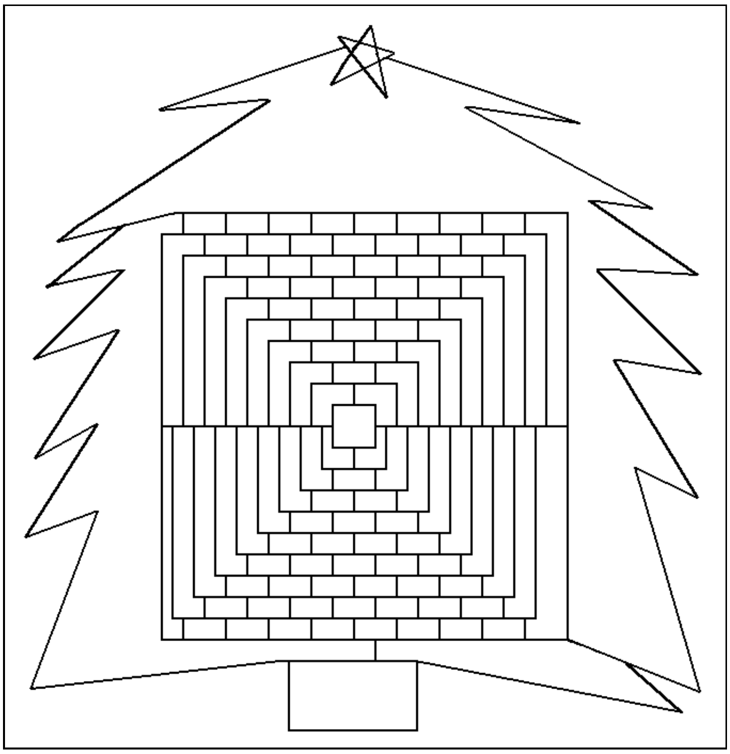
$$(a \wedge c) \Rightarrow (b \Rightarrow (d \Rightarrow c)).$$

   What does Harrison's `pinterpolate` function return?

Randomized k-satisfiability and phase-transition. Implement a function to generate all propositional formulas with a given number k of symbols. Plot the proportion $\frac{taut(k)+contr(k)}{for(k)}$ of such formulas that are tautologies or contradictions against their size k. Can you generate results for large enough lengths to see a trend? Is the trend as expected?

[1] Or a consequence, due to the "deduction theorem", i.e., the property that states that $\models \varphi \Rightarrow \psi$ iff $\varphi \models \psi$.

April's Fool. This exercise is a follow-up of Lab III, in particular, the 4-colorability exercise. Gardner in 1975 gave a planar map which he claimed (as an April Fool's joke) not to be 4-colorable:



This clearly contradicts the 4-color theorem of graph theory, that states that a 4-coloring exists for *every* planar graph.

Encode Gardner's map into a propositional formula/set of clauses and refute the claim by proving it satisfiable, using, e.g., Minisat or any other solver (hint: you will have to encode the colorability conditions too!).